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Some Identities Between Wely's Curvature Tensor and Conformal Curvature Tensor

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Abstract

In this paper, we investigate some identities between Weyl's curvature tensor and conformal curvature tensor C^i_{jkh} . We first introduce the basic concepts of Weyl's tensor W^i_{jkh} and conformal tensor C^i_{jkh}. Then, we derive some identities between these two tensors. Finally, we apply these identities to some examples.

Keywords: Covariant Derivative of Fourth Orders, Weyl Tensor W^i_{jkh} , Conformal Tensor C^i_{jkh} and Cartan's 3th Curvature Tensor R^i_{jkh} .

1. Introduction

Weyl's tensor W^i_{jkh} and conformal tensor C^i_{jkh} are two important geometric objects in differential geometry. They are both used to study the curvature of spacetime. Weyl's tensor W^i_{jkh} is a conformal invariant, which means that it is invariant under conformal transformations. The conformal tensor C^i_{jkh} is not a conformal invariant, but it is related to Weyl's tensor W^i_{jkh} by a simple formula.

In this paper, we investigate some identities between Weyl's tensor W^i_{jkh} and conformal tensor C^i_{jkh} . We first introduce the basic concepts of Weyl's curvature tensor and conformal curvature tensor. Then, we derive some identities between these two tensors. Finally, we apply these identities to some examples. The concept of the three-dimensional of Riemannian space with recurrent curvature was studied and explored by Rund (1981).

In the context of recurrent Finsler spaces, the analysis of generalized curvature tensors relies on the Berwald curvature tensor, which has been discussed by Abdallah (2017), AL-Qashbari (2020), and others (see references). Properties of the curvature tensor W^i_{jkh} were investigated by Ahsan & Ali (2014), Hadi (2016), Al-Qashbari and Qasem (2017), Abu-Donia, Shenawy and Abdelhameed (2020), and others.

A Generalized bi-recurrent, tri-recurrent in Finsler spaces, and higher-orders recurrent spaces have been studied (see references). Additionally, Ahsan and Ali (2014) studied some of curvature tensors in the space-time for general relativity. Pandey, Saxena, and Goswani (2011) investigated complete Finsler spaces of a generalized H-recurrent space.

Decomposability of some tensors in recurrent Finsler spaces has been a topic of research by Al-Qashbari (2020), Al-Qashbari and AL-Maisary (2023) and others.

The metric tensor g_{ij} and $\mathcal{B}_k$ (Berwald's connection coefficients) G^i_{jk} are positively homogeneous of degree 0 in directional arguments.

Two vectors y_i and y^i meet the following conditions

a) $y_i = g_{ij}\, y^j$ and b) $y_i\, y^i = F^2$, c) $\delta^k_j y^j = y^k$ (1.1)

The quantities g_{ij} and g^{ij} are related by [11]

$$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 \;, & if \quad i = k \;, \\ 0 \;, & if \quad i \neq k \;. \end{cases} \qquad (1.2)$$

Tensor C_{ijk} is known as (h)hv-torsion tensor defined by

$$C_{ijk} = \tfrac{1}{2}\dot{\partial}_i\, g_{jk} = \tfrac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2. \qquad (1.3)$$

The (v)hv-torsion tensor C^h_{ik} and tensor C_{ijk} are given by

a) $C^i_{jk} y^j = C^i_{jk} y^k = 0$ and

b) $C_{ijk}\, y^i = C_{ijk}\, y^j = C_{ijk}\, y^k = 0.$ (1.4)

Covariant derivative $\mathcal{B}_k T^i_j$ for Berwald's $(\mathcal{B}_k)$ of any tensor T^i_j w. r. t. x^k is defined as

$$\mathcal{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)\, G^r_k \;\; + T^r_j G^i_{rk} - T^i_r G^r_{jk}. \qquad (1.5)$$

The vector y^i and metric function F are vanished identically for Berwald's covariant derivative.

a) $\mathcal{B}_k F = 0$ and b) $\mathcal{B}_k y^i = 0$. (1,6)

Metric tensor g_{ij} is not equal to zero (i.e. not vanish), defined by

$$\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h \mathcal{B}_h C_{ijk}. \qquad (1.7)$$

Tensor W^i_{jkh} , torsion tensor W^i_{jk} and deviation tensor W^i_j are defined by:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\big(n\, H_{jh} + H_{hj} + y^r\dot{\partial}_j H_{hr} - \frac{\delta^i_h}{(n^2-1)}\big(n\, H_{jk} + H_{kj} + y^r\dot{\partial}_j H_{kr}\big), \qquad (1.8)$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\,\Big\{ \frac{\delta^i_{[j}}{(n^2-1)}\big(n\, H_{k]} - y^r H_{k]\,r}\big) \Big\} \qquad (1.9)$$

and

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{(n+1)}\big(\dot{\partial}_r H^r_j - \dot{\partial}_j H\big)\, y^i, \qquad (1,10)$$

respectively.

The tensors W^i_{jkh} and W^i_{jk} give the following identities

a) $W^i_{jkh}\, y^j = W^i_{kh}$ and b) $W^i_{jk}\, y^j = W^i_{k.}$ (1.11)

Also, if we suppose that the tensor W^i_j and W satisfy the following identities

a) $W^i_{ki} = W_k$ and b) $W^i_i = W.$ (1.12)

The skew-symmetric in its indices k and h in the tensor W^i_{jkh} .

Cartan's 3[th] curvature tensor R^i_{jkh}, Ricci tensor R_{jk}, the vector H_k and scalar curvature H are defined as

a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + (\Gamma^{*i}_{ljk})\, G^l_h + C^i_{jm}(G^m_{kh} - G^m_{kl}G^l_h) + \Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} - \Gamma^{*i}_{kjh} - (\Gamma^{*i}_{ljh})\, G^l_k - C^i_{jm}(G^m_{hk} - G^m_{hl}G^l_k) + \Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}$,

b) $R^i_{jkh}\, y^j = H^i_{kh}$, c) $R_{jk}\, y^j = H_k$,

d) $R_{jk}\, y^k = R_j$, h) $R^i_i = R$,

e) $R^i_{jki} = R_{jk}$ and f) $H_i\, y^i = H^i_i = (n-1)\, H.$ (1.13)

AL-Qashbari, and AL-Maisary have studied the generalized W^i_{jkh} of fourth orders recurrent in Finsler space and they found it the following equation

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W_{jrkh} = a_{lmns} C_{jrkh} + b_{lmns}(g_{rh}g_{jk} - g_{rk}g_{jh}) - 2b_{lmn}\,\mathcal{B}_r y^r(\delta^i_h C_{jks} - \delta^i_k C_{jhs})g_{ir}$$
$$- 2\mathcal{B}_s b_{lm}\,\mathcal{B}_q y^q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})g_{ir} - 2b_{lm}\mathcal{B}_q y^q\mathcal{B}_s(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})g_{ir}\, 2b_{lm}\mathcal{B}_q y^q\mathcal{B}_s$$
$$- 2\mathcal{B}_s\mathcal{B}_n b_l\mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir} - 2\,\mathcal{B}_n b_l\mathcal{B}_p y^p\mathcal{B}_s(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir} - 2\mathcal{B}_s b_l\mathcal{B}_p y^p\mathcal{B}_n(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir} -$$
$$2b_l\mathcal{B}_q\, y^q\,\mathcal{B}_s\,\mathcal{B}_n(\,\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\,)\, g_{ir} - 2\,(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\,\mathcal{B}_t\, y^t C_{irl})W^i_{jkh} - 2\,[\,a_s\, W^i_{jkh} + \, b_s(\,\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,)\,] - \, 2(\mathcal{B}_s\mathcal{B}_m\,\mathcal{B}_t\, y^t\, C_{irl}\,)$$
$$[\,a_n\, W^i_{jkh} + \, b_n(\,\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,)\,]$$
$$- \, 2(\,\mathcal{B}_m\,\mathcal{B}_t\, y^t\, C_{irl}\,)[\,a_{ns}\, W^i_{jkh} \, + b_{ns}(\,\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2b_s\mathcal{B}_p\, y^p(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,)] - \, 2$$
$$(\,\mathcal{B}_s\,\mathcal{B}_n\,\mathcal{B}_t\, y^t\, C_{irl}\,)\,[\,a_m\, W^i_{jkh} + b_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}\,)]$$
$$- 2(\mathcal{B}_n\mathcal{B}_t\, y^t\, C_{irl})[\,a_{ms}\, W^i_{jkh} + b_{ms}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p\, y^p(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\,)]$$
$$-2(\,\mathcal{B}_s\,\mathcal{B}_t\, y^t\, C_{irl})[\,a_{mn}\, W^i_{jkh} + b_{mn}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2b_n\mathcal{B}_p\, y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\,)]$$
$$- 2(\,\mathcal{B}_t\, y^t\, C_{irl}\,)[\,a_{mns}\, W^i_{jkh} + b_{mns}\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2b_{mn}\,\mathcal{B}_q\, y^q\,(\delta^i_h\, C_{jks} - \delta^i_k\, C_{jhs}) - \, 2\,\mathcal{B}_s b_m\,\mathcal{B}_p\, y^p\,(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}) - 2b_m\,\mathcal{B}_p\, y^p\,\mathcal{B}_s(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn})]$$
$$- 2(\mathcal{B}_s\,\mathcal{B}_n\,\mathcal{B}_t\, y^t\, C_{irm}\,)[a_l\, W^i_{jkh} + b_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})]$$
$$- 2(\,\mathcal{B}_n\,\mathcal{B}_t\, y^t C_{irm})[a_{ls}\, W^i_{jkh} + b_{ls}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p\, y^p(\delta^i_h\, C_{jkl} - \delta^i_k\, C_{jhl}\,)] - \, 2$$
$$(\,\mathcal{B}_s\,\mathcal{B}_t\, y^t\, C_{irm})[\,a_{ln}\, W^i_{jkh} + b_{ln}\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) - 2b_n\mathcal{B}_p\, y^p(\delta^i_h\, C_{jkl} - \delta^i_k\, C_{jhl}\,)]$$
$$- 2(\,\mathcal{B}_t\, y^t\, C_{irm}\,)[\,a_{lns}\, W^i_{jkh} \, + b_{lns}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2b_{ln}\,\mathcal{B}_q\, y^q\,(\delta^i_h\, C_{jks} - \delta^i_k\, C_{jhs})$$
$$- 2\mathcal{B}_s b_l\,\mathcal{B}_p\, y^p\,(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}) - 2b_l\,\mathcal{B}_p\, y^p\,\mathcal{B}_s(\,\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn})]$$

$-2(\mathcal{B}_s\,\mathcal{B}_t\,y^t\,C_{irn})[\,a_{lm}\,W^i_{jkh}+b_{lm}\,(\delta^i_h\,g_{jk}-\delta^i_k\,g_{jh})-2b_m\mathcal{B}_p\,y^p(\delta^i_h\,C_{jkl}-\delta^i_k\,C_{jhl})\,]-2(\mathcal{B}_t\,y^t\,C_{irn})-2\,b_{ms}\,\mathcal{B}_q\,y^q(\delta^i_h\,C_{jkl}-\delta^i_k\,C_{jhl})$
$-2\mathcal{B}_l b_s\,\mathcal{B}_p\,y^p\,(\delta^i_h\,C_{jkm}-\delta^i_k\,C_{jhm})-2\,b_m\,\mathcal{B}_p\,y^p\,\mathcal{B}_l(\delta^i_h\,C_{jks}-\delta^i_k\,C_{jhs})]$
$-2(\mathcal{B}_t\,y^t\,C_{irs})[a_{lmn}\,W^i_{jkh}+b_{lmn}\,(\delta^i_h\,g_{jk}-\delta^i_k\,g_{jh})-2b_{lm}\,\mathcal{B}_q\,y^q(\delta^i_h\,C_{jkn}-\delta^i_k\,C_{jhn})$
$-2\mathcal{B}_n b_l\,\mathcal{B}_p\,y^p(\delta^i_h C_{jkm}-\delta^i_k C_{jhm})-2\,b_l\,\mathcal{B}_p y^p\,\mathcal{B}_n(\delta^i_h C_{jkm}-\delta^i_k C_{jhm})]$. (1.14)

2. Generalized –BW-Four- Recurrent Space

Our work in this paper we defined $\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$ is covariant derivative of 4th order, which is defined as

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l W^i_{jkh}=a_{lmns}\,W^i_{jkh}+b_{lmns}(\delta^i_h g_{jk}-\delta^i_k g_{jh})-2b_{lmn}\,\mathcal{B}_q(\delta^i_h C_{jks}-\delta^i_k C_{jhs})\,y^q-2\,b_{lms}\mathcal{B}_q(\delta^i_h C_{jkn}-\delta^i_k C_{jhn})\,y^q-2\,b_{lm}\,\mathcal{B}_s\mathcal{B}_q(\delta^i_h C_{jkn}-\delta^i_k C_{jhn})\,y^q-2\,b_{lns}\,\mathcal{B}_p(\,\delta^i_h C_{jkm}-\delta^i_k C_{jhm})\,y^q-2\,b_{ln}\mathcal{B}_s\mathcal{B}_q(\,\delta^i_h C_{jkm}-\delta^i_k C_{jhm})\,y^q-2b_{ls}\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm}-\delta^i_k C_{jhm})\,y^q-2\,b_l\mathcal{B}_s\mathcal{B}_n\mathcal{B}_q(\delta^i_h C_{jkm}-\delta^i_k C_{jhm})\,y^q. \quad (2.1)$$

where $\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$ is covariant differential operator of Berwald's of 4th orders w. r. t. x^l, x^m, x^n and x^s, respectively, such that a_{lmns} and b_{lmns} are not equal to zero covariant vector fields.

Result 2.1. All a generalized $\mathcal{B}W$-recurrent space is a generalized $\mathcal{B}W$-fourth recurrent space.

Definition 2. 1. A Finsler space of tensor W^i_{jkh} is called as projective curvature tensor and is known as satisfies (2.1), will be called a generalized-four recurrent space. We shall call this Finsler space as a generalized $\mathcal{B}W$-fourth-recurrent space and we denoted by G$\mathcal{B}$W-FRF$_n$.

Transvecting condition to a higher dimensional space (2.1) by y^j, using (1.1a), (1.4b), (1.6b) and (1.11a), we get

$$\mathcal{B}_s\mathcal{B}_n\,\mathcal{B}_m\mathcal{B}_l\,W^i_{kh}=a_{lmns}\,W^i_{kh}+b_{lmns}\,(\,\delta^i_h\,y_k-\delta^i_k\,y_h\,). \quad (2.2)$$

Again, transvecting condition to a higher dimensional space (2.2) by y^k, using (1.1b), (1.2), (1.6b) and (1.11b), we get

$$\mathcal{B}_n\,\mathcal{B}_m\mathcal{B}_l\,W^i_h=a_{lmns}\,W^i_h+b_{lmns}\,(\,\delta^i_h F^2-y_h\,y^i\,). \quad (2.3)$$

Therefore, the proof of theorem is completed, we can say

Theorem 2.1. In GBW-FRF$_n$, covariant derivative for Berwald of fourth order for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by (2.2) and (2.3).

Contracting the index space by summing over i and h in the conditions (2.2) and (2.3), using (1.1b), (1.2), (1.12a) and (1.12b), we get

$$\mathcal{B}_n\,\mathcal{B}_m\mathcal{B}_l\,W_k=a_{lmns}\,W_k+(n-1)\,b_{lmns}\,y_k \quad (2.4)$$

and

$$\mathcal{B}_s\mathcal{B}_n\,\mathcal{B}_m\mathcal{B}_l\,W=a_{lmns}\,W+(n-1)\,b_{lmns}\,F^2. \quad (2.5)$$

From conditions (2.4) and (2.5), we show that the curvature vector W_k and the curvature scalar W cannot equal to zero because if the vanishing of any one of these would imply $a_{lmns}=0$ and $b_{lmns}=0$, that is a contradiction.

So the proof of theorem is completed, we can say

Theorem 2.2. In GBW-FRF$_n$, the vector W_k and the scalar W in equations (2.4) and (2.5), are non-vanishing, respectively.

3. Wely's Curvature Tensor and Conformal Curvature Tensor

Some properties of W^i_{jkh} curvature tensor were proposed by Ahsan and Ali [1] in (2014).

For $(n=4)$ a Riemannian space, Weyl defined the conformal curvature tensor C^i_{jkh} often known as the Weyl conformal curvature tensor, as

$$W^i_{jkh}=\mathrm{C}^i_{jkh}+\tfrac{1}{2}(g_{jh}R^i_k-\delta^i_h\,R_{jk})+\tfrac{5}{6}(\delta^i_k\,R_{jh}-g_{jk}\,R^i_h)+\tfrac{R}{6}(\,\delta^i_h\,g_{jk}-\delta^i_k\,g_{jh})\,. \quad (3.1)$$

The curvature tensor C^i_{jkh}, torsion tensor C^i_{jk} , Ricci tensor C_{jk}, curvature vector C_k and scalar curvature C are satisfying:

a) $\mathrm{C}^i_{jkh}\,y^j=\mathrm{C}^i_{kh}$, b) $\mathrm{C}^i_{kh}\,y^k=\mathrm{C}^i_h$, c) $\mathrm{C}^i_{jki}=\mathrm{C}_{jk}$

d) $\mathrm{C}^i_{ki}=\mathrm{C}_k$ and e) $\mathrm{C}^i_i=\mathrm{C}$. (3.2)

The divergence of W^i_{jkh} projective curvature tensor in terms four orders (Berwald's covariant derivative) of (3.1), w. r. t. x^l , x^m , x^n and x^s, successively, of projective curvature tensor may be written as

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,W^i_{jkh}=\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,C^i_{jkh}+\tfrac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(g_{jh}R^i_k-\delta^i_h\,R_{jk})+\tfrac{5}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\delta^i_k\,R_{jh}-g_{jk}\,R^i_h\,)+\tfrac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\,\delta^i_h\,g_{jk}-\delta^i_k\,g_{jh}). \quad (3.3)$$

So that from equations (2.1) and (3.3), we get

$$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,C^i_{jkh}+\tfrac{1}{2}\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\,g_{jh}\,R^i_k-\delta^i_h\,R_{jk}\,)+\tfrac{5}{6}\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\,\delta^i_k\,R_{jh}-g_{jk}\,R^i_h\,)+\tfrac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\,\delta^i_h\,g_{jk}-\delta^i_k\,g_{jh})=a_{lmns}\left(C^i_{jkh}+\tfrac{1}{2}(g_{jh}R^i_k-\delta^i_hR_{jk})+\tfrac{5}{6}(\delta^i_kR_{jh}-\,g_{jk}R^i_h)+\tfrac{R}{6}(\,\delta^i_h\,g_{jk}-\delta^i_k\,g_{jh})\right)+b_{lmns}(\,\delta^i_h\,g_{jk}-\delta^i_k\,g_{jh}\,)-2\,b_{lmn}\,\mathcal{B}_r\,y^r(\delta^i_h\,C_{jks}-\delta^i_k\,C_{jhs})-2\,\mathcal{B}_s b_{lm}\,\mathcal{B}_q\,y^q(\,\delta^i_h\,C_{jkn}-\delta^i_k\,C_{jhn}\,)-2\,b_{lm}\,\mathcal{B}_q\,y^q\mathcal{B}_s(\delta^i_h\,C_{jkn}-\delta^i_k\,C_{jhn}\,)-2\,\mathcal{B}_s\mathcal{B}_n b_l\mathcal{B}_p y^p(\delta^i_h\,C_{jkm}-\delta^i_k\,C_{jhm})-2\,\mathcal{B}_n b_l\mathcal{B}_p y^p\mathcal{B}_s(\delta^i_h\,C_{jkm}-\delta^i_k\,C_{jhm})-2\,\mathcal{B}_s b_l\mathcal{B}_p y^p\mathcal{B}_n(\delta^i_h\,C_{jkm}-\delta^i_k\,C_{jhm})-2\,b_l\mathcal{B}_q\,y^q\,\mathcal{B}_s\,\mathcal{B}_n(\,\delta^i_h\,C_{jkm}-\delta^i_k\,C_{jhm}\,). \quad (3.4)$$

Which can be written as

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C^i_{jkh} = a_{lmns}\, C^i_{jkh} + \frac{1}{2}\, a_{lmns}\big(\, g_{jh}\, R^i_k\, - \delta^i_h\, R_{jk}\,\big) + \frac{5}{6}\, a_{lmns}\, (\delta^i_k\, R_{jh} - g_{jk}\, R^i_h) + \frac{R}{6} a_{lmns}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})\; + b_{lmns}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - b_{lmn}\, \mathcal{B}_r\, y^r\big(\delta^i_h\, C_{jks} - \delta^i_k\, C_{jhs}\,\big) - 2\, \mathcal{B}_s b_{lm}\, \mathcal{B}_q\, y^q\big(\, \delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,\big) - 2\, b_{lm}\, \mathcal{B}_q\, y^q \mathcal{B}_s\big(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\big) - 2\mathcal{B}_s\mathcal{B}_n b_l \mathcal{B}_p y^p \big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\, \mathcal{B}_n b_l \mathcal{B}_p y^p \mathcal{B}_s\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\mathcal{B}_s b_l \mathcal{B}_p y^p \mathcal{B}_n\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\, b_l\mathcal{B}_q\, y^q \mathcal{B}_s\, \mathcal{B}_n\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - \frac{1}{2}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m \mathcal{B}_l\,\big(\, g_{jh}\, R^i_k\, - \delta^i_h\, R_{jk}\big) - \frac{5}{6}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, \delta^i_k\, R_{jh} - g_{jk}\, R^i_h\,\big) - \frac{R}{6}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}).$ (3.5)

Or, we can write as

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C^i_{jkh} = a_{lmns}\, C^i_{jkh} + \; b_{lmns}\,\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - 2\, b_{lmn}\, \mathcal{B}_r\; y^r\big(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\big) - \; 2\, \mathcal{B}_s b_{lm}\, \mathcal{B}_q\; y^q \big(\, \delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,\big) - 2\, b_{lm}\, \mathcal{B}_q\; y^q \mathcal{B}_s\big(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,\big) - 2\mathcal{B}_s\mathcal{B}_n b_l \mathcal{B}_p y^p\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\, \mathcal{B}_n b_l \mathcal{B}_p y^p \mathcal{B}_s\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\mathcal{B}_s b_l \mathcal{B}_p y^p \mathcal{B}_n\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\, b_l\mathcal{B}_q\; y^q \mathcal{B}_s\, \mathcal{B}_n\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\,\big) - \frac{1}{2}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, g_{jh} R^i_k - \delta^i_h\, R_{jk}\big) - \frac{5}{6}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, \delta^i_k\, R_{ji} - g_{jk}\, R^i_h\,\big) - \frac{R}{6}\, \mathcal{B}_s\mathcal{B}_n\,\,(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) + \frac{1}{2}\, a_{lmns}\,\big(\, g_{jh} R^i_k - \delta^i_h\, R_{jk}\big) + \frac{5}{6}\, a_{lmns}\big(\, \delta^i_k\, R_{jh} - g_{jk}\, R^i_h\,\big) + \frac{R}{6}\, a_{lmns}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})\,.$ (3.6)

This shows that

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C^i_{jkh} = a_{lmns}\, C^i_{jkh} + \; b_{lmns}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - 2\, b_{lmn}\, \mathcal{B}_r y^r\big(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\big) - \; 2\, \mathcal{B}_s\, b_{lm}\, \mathcal{B}_q\; y^q\big(\, \delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,\big) - 2\, b_{lm}\, \mathcal{B}_q\; y^q \mathcal{B}_s\big(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,\big) - 2\, \mathcal{B}_s\mathcal{B}_n b_l \mathcal{B}_p y^p\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\, \mathcal{B}_n b_l \mathcal{B}_p y^p \mathcal{B}_s\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\mathcal{B}_s b_l \mathcal{B}_p y^p \mathcal{B}_n\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big) - 2\, b_l\mathcal{B}_q y^q \mathcal{B}_s \mathcal{B}_n\big(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\big)\;\; .$ (3.7)

If and only if

$\frac{1}{2}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, g_{jh} R^i_k\, - \delta^i_h\, R_{jk}\big) + \frac{5}{6}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, \delta^i_k\, R_{jh} - g_{jk}\, R^i_h\,\big) + \frac{R}{6}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) = \frac{1}{2}\, a_{lmns}\,\,\big(\, g_{jh} R^i_k\, - \delta^i_h\, R_{jk}\big) + \frac{R}{6}\, a_{lmns}(\, \delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}).$ (3.8)

In conclusion the proof of theorem is completed, we can determine

Theorem 3.1. In GBW-FRF$_n$, conformal curvature tensor C^i_{jkh} is a generalized fourth recurrent Finsler space if the condition (3.8) holds good.

Transvecting condition (3.6) by y^j, using (3.2a), (1.6b), (1.1a), (1.4b) and (1.13c), we get

$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C^i_{kh} = a_{lmns}\, C^i_{kh} + b_{lmns}\,(\, \delta^i_h\, y_k - \delta^i_k\, y_h\,) - \frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\, y_h\, R^i_k\, - \delta^i_h\, H_k) - \frac{5}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, (\delta^i_k H_h - \; y_k R^i_h) - \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\delta^i_h\, y_k - \delta^i_k\, y_h\big) + \frac{1}{2} a_{lmns}\big(y_h R^i_k - \delta^i_h H_k\big) + \; \frac{5}{6}\; a_{lmns}\,(\, \delta^i_k\, H_h - \; y_k\, R^i_h\,) + \frac{R}{6} a_{lmns}(\, \delta^i_h\, y_k - \delta^i_k\, y_h)\;.$ (3.9)

This shows that

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C^i_{kh} = a_{lmns}\, C^i_{kh} + b_{lmns}\,(\, \delta^i_h\, y_k - \delta^i_k\, y_h\,).$ (3.10)

If and only if

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\, y_h\, R^i_k\, - \delta^i_h\, H_k) + \frac{5}{3}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\, \delta^i_k\, H_h - y_k\, R^i_h\,) + \frac{R}{3}\, \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\, \delta^i_h\, y_k - \delta^i_k\, y_h) = a_{lmns}(\, y_h\, R^i_k - \delta^i_h\, H_k) + \frac{5}{3}\, a_{lmns}\,(\, \delta^i_k\, H_h - y_k\, R^i_h\,) + \frac{R}{3}\, a_{lmns}(\delta^i_h\, y_k - \delta^i_k\, y_h)\;.$ (3.11)

The proof of theorem is completed, we conclude

Theorem 3.2. In GBW-FRF$_n$, the covariant derivative of the fourth orders for the torsion tensor C^i_{kh} (Conformal curvature tensor C^i_{jkh}) is a generalized 4th recurrent Finsler space if the condition (3.11) holds good.

Transvecting (3.9) by y^k, using $(n = 4)$, (3.2b), (1.6b), (1.1b), (1.1c) and (1.13f), we get

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C^i_h = a_{lmns} C^i_h + b_{lmns}\big(\delta^i_h F^2 - y_h\, y^i\big) - \frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\big(\, y_h\, R^i_k\, y^k - 3\, \delta^i_h H\big) - \frac{5}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, y^i H_h - F^2 R^i_h\,\big) - \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, \delta^i_h\, F^2 - y_h\, y^i\big) + \frac{1}{2} a_{lmns}\big(y_h\, R^i_k\, y^k - 3\, \delta^i_h H\big) + \frac{5}{6} a_{lmns}\big(y^i H_h - F^2 R^i_h\big) + \frac{R}{6} a_{lmns}\,\big(\delta^i_h\, F^2 - y_h\, y^i\big).$ (3.12)

This shows that

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C^i_h = a_{lmns}\, C^i_h + b_{lmns}\,\big(\delta^i_h F^2 - y_h\, y^k\big)\,.$ (3.13)

If and only if

$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\delta^k_h\;\, R^i_k\; - 3\, \delta^i_h\, H\,\big) = a_{lmns}\,\big(\delta^k_h\;\, R^i_k\; - 3\, \delta^i_h\, H\,\big)\,;$

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,\big(\, y^i H_h - F^2\, R^i_h\,\big) = a_{lmns}\,\big(\, y^i H_h - F^2 R^i_h\big)$ (3.14)

and

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, R\big(\delta^i_h\, F^2 - y_h\, y^i\big) = a_{lmns} R(\delta^i_h\, F^2 - y_h\, y^i).$

Thus, the proof of theorem is completed, we conclude

Theorem 3.3. In GBW-FRF$_n$, the covariant derivative of the fourth orders for the projective deviation tensor C^i_h is a generalized 4th recurrent Finsler space if the tensors $\big(\delta^k_h R^i_k - (n-1)\delta^i_h H\big)$, $\big(\, y^i H_h - F^2 R^i_h\,\big)$ and $R(\delta^i_h F^2 - y_h y^i)$ are recurrent Finsler space of fourth orders.

Contracting the indices i and h in the equations (3.6), (3.9) and (3.12), using (3.2c), (3.2d), (3.2e), (1.2), (1.1a), (1.1b), (1.13h) and (1.1c), in view of (3.2c), (3.2d) and (3.2e), we get

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C_{jk} = a_{lmns}\, C_{jk} + (n-1) b_{lmns}\, g_{jk}$$

$-2(n-1)\, b_{lmn}\, \mathcal{B}_r y^r C_{jks} - 2\,(n-1)\mathcal{B}_s\, b_{lm}\, \mathcal{B}_q\, y^q C_{jkn} - 2\,(n-1) b_{lm}\, \mathcal{B}_q\, y^q\, \mathcal{B}_s C_{jkn} - 2(n-1)\mathcal{B}_n b_l \mathcal{B}_p\, y^p \mathcal{B}_s C_{jkm} - 2\,(n-1)\mathcal{B}_n b_l\, \mathcal{B}_p y^p \mathcal{B}_s C_{jkm} - 2(n-1)\mathcal{B}_s b_l\, \mathcal{B}_p y^p \mathcal{B}_n C_{jkm} - 2(n-1) b_l\, \mathcal{B}_q\, y^q\, \mathcal{B}_s \mathcal{B}_n C_{jkm} - \frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{ji}R^i_k - n\, R_{jk}\right) - \frac{5}{6}\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(R_{jk} - g_{jk}\, R\right) - \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(n-1)\, g_{jk} + \frac{1}{2} a_{lmns}\left(g_{ji}R^i_k - n\, R_{jk}\right) + \frac{5}{6} a_{lmns}\left(R_{jk} - g_{jk}R\right) + \frac{R}{6}(n-1)\, a_{lmns}\, g_{jk}\,.$ (3.15)

This shows that

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C_{jk} = a_{lmns}\, C_{jk} + (n-1) b_{lmns}\, g_{jk} - 2(n-1)\, b_{lmn}\, \mathcal{B}_r y^r C_{jks} - 2(n-1)\mathcal{B}_s\, b_{lm}\, \mathcal{B}_q\, y^q C_{jkn} - 2(n-1) b_{lm}\, \mathcal{B}_q\, y^q\, \mathcal{B}_s C_{jkn} - 2(n-1)\mathcal{B}_n b_l \mathcal{B}_p\, y^p C_{jkm} - 2(n-1)\mathcal{B}_n b_l\, \mathcal{B}_p\, y^p\, \mathcal{B}_s - 2(n-1)\, \mathcal{B}_s b_l\, \mathcal{B}_p\, y^p\, \mathcal{B}_n C_{jkm} - 2(n-1)\, b_l\, \mathcal{B}_q\, y^q\, \mathcal{B}_s\, \mathcal{B}_n C_{jkm}\,.$ (3.16)

If and only if

$\frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{jh}R^h_k - n\, R_{jk}\right) + \frac{5}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(R_{jk} - g_{jk}\, R\right) + \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(n-1)\, g_{jk} = \frac{1}{2} a_{lmns}\left(g_{ji}R^i_k - n\, R_{jk}\right) + \frac{5}{6} a_{lmns}\left(R_{jk} - g_{jk}R\right) + \frac{R}{6}(n-1)\, a_{lmns}\, g_{jk}$ (3.17)

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C_k = a_{lmns}\, C_k + (n-1) b_{lmns}\, y_k - \frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y_i R^i_k - n\, H_k\right) - \frac{5}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(H_k - y_k\, R) - \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(n-1)\, y_k + \frac{1}{2} a_{lmns}\left(y_i R^i_k - n\, H_k\right) + \frac{5}{6} a_{\,lmns}(H_k - y_k\, R) + \frac{R}{6}(n-1)\, a_{lmns}\, y_k\,.$ (3.18)

This shows that

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C_k = a_{lmns}\, C_k + (n-1)\, b_{lmns}\, y_k\,.$ (3.19)

If and only if

$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y_h R^h_k - n\, H_k\right) + \frac{5}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(H_k - y_k\, R) + \frac{R}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(n-1)\, y_k = a_{lmns}\left(y_h\, R^h_k - n\, H_k\right) + \frac{5}{3} a_{lmns}(H_k - y_k\, R) + \frac{R}{3}(n-1)\, a_{lmns}\, y_k\,.$ (3.20)

In the last

$\mathcal{B}_m\mathcal{B}_l\, C = a_{lmns}\, C + b_{lmns}(n-1)F^2 - \frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y_i\, R^i_k\, y^k - 3nH\right) - \frac{5}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y^i H_i - F^2 R\right) - (n-1)\frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l + \frac{1}{2} a_{lmns}\left(y_i\, R^i_k\, y^k - 3nH\right) + \frac{5}{6} a_{lmns}\left(y^i H_i - F^2 R\right) + (n-1)\frac{R}{6} a_{lmns}\, F^2\,.$ (3.21)

This shows that

$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, C = a_{lmns}\, C + + b_{lmns}(n-1)F^2\,.$ (3.22)

If and only if

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y_i\, R^i_k\, y^k - 3nH\right) + \frac{5}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(y^i H_i - F^2 R\right) + (n-1)\frac{R}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, F^2 = a_{lmns}\left(y_i\, R^i_k\, y^k - 3nH\right) + \frac{5}{3} a_{lmns}\left(y^i H_i - F^2 R\right) + (n-1)\frac{R}{3} a_{lmns}\, F^2.$ (3.23)

In conclusion the proof of theorem is completed, we get

Theorem 3.4. In GBW-FRF$_n$, Ricci tensor C_{jk} , vector C_k and scalar C are given in (3.16), (3.19) and (3.22) if and only if the conditions (3.17), (3.20) and (3.23) are holds good, respectively.

The tensor C_{jrkh} for a V_4 by the formula [1]; is defined as

$$R_{jrkh} = C_{jrkh} + \frac{1}{2}\left(g_{jk}R_{rh} + g_{rh}\, R_{jk} + g_{jh}R_{jk} - g_{jh}\, R_{rk} - g_{rk}\, R_{jh}\right) + \frac{R}{6}\left(g_{jh}\, g_{rk} - g_{jk}\, g_{rh}\right). \quad (3.24)$$

(3.24)

It is known that the associate tensor Cartan's third tensor R_{jrkh} and the associate tensor W_{ijkh} are connected by

$$W_{jrkh} = R_{jrkh} + \frac{1}{3}\left(g_{jk}\, R_{rh} - g_{rk}R_{jh}\right). \quad (3.25)$$

Compensate for the relationship (3.24) in (3.25), we get

$$W_{jrkh} = C_{jrkh} + \frac{1}{2}\left(g_{jk}R_{rh} + g_{rh}\, R_{jk} + g_{jh}R_{jk} - g_{jh}\, R_{rk} - g_{rk}\, R_{jh}\right)\left(g_{jh}\, g_{rk} - g_{jk}\, g_{rh}\right) + \frac{1}{3}\left(g_{jk}\ R_{rh} - g_{rk}\, R_{jh}\right)\,. \quad (3.26)$$

Taking the covariant derivative of four orders of (3.26), w. r. t. x^l , x^m , x^n and x^s, successively, we get the following condition:

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\, W_{jrkh} = \mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C_{jrkh} + \frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{jk}R_{rh} + g_{rh}R_{jk} + g_{jh}R_{jk} - g_{jh}\, R_{rk} - g_{rk}\, R_{jh}\right) + \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{jh}\, g_{rk} - g_{jk}\, g_{rh}\right) + \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{jk}R_{rh} - g_{rk}R_{jh}\right)\,. \quad (3.27)$$

Using the conditions (1.14) and (3.26) in (3.27), we get

$$+\frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{jk}\, R_{rh} + g_{rh}\, R_{jk} + g_{jh}\, R_{jk} - g_{jh}\, R_{rk} - g_{rk}\, R_{jh}\right) + \frac{R}{6}\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{jh}\, g_{rk} - g_{jk}\, g_{rh}\right) + \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\left(g_{jk}R_{rh} - g_{rk}R_{jh}\right)$$
$$= a_{lmns}\Big(C_{jrkh} + \frac{1}{2}\left(g_{jk}R_{rh} + g_{rh}R_{jk} + g_{jh}R_{jk} - g_{jh}R_{rk} - g_{rk}R_{jh}\right) + \frac{R}{6}\left(g_{jh}g_{rk} - g_{jk}g_{rh}\right) \frac{1}{3}\left(g_{jk}\, R_{rh} - g_{rk}\, R_{jh}\right)\Big) + b_{lmns}\left(g_{rh}\, g_{jk} - g_{rk}\, g_{jh}\right) - 2b_{lmn}\, \mathcal{B}_r y^r\left(\delta^i_h\, C_{jks} - \delta^i_k C_{jhs}\right) g_{ir} - 2\mathcal{B}_s b_{lm}\, \mathcal{B}_q y^q \left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) g_{ir} - 2b_{lm}\mathcal{B}_q y^q \mathcal{B}_s\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) g_{ir} - 2\mathcal{B}_s\mathcal{B}_n b_l \mathcal{B}_p y^p\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) g_{ir} - 2\,\mathcal{B}_n b_l \mathcal{B}_p y^p \mathcal{B}_s\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) g_{ir} - 2\mathcal{B}_s b_l \mathcal{B}_p y^p \mathcal{B}_n \left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) g_{ir} - 2b_l \mathcal{B}_q\, y^q\, \mathcal{B}_s\, \mathcal{B}_n\left(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\right) g_{ir} - 2(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\, \mathcal{B}_t\, y^t C_{irl}) W^i_{jkh} - 2(\,\mathcal{B}_n\mathcal{B}_m\, \mathcal{B}_t\, y^t\, C_{irl})\left[\, a_s\, W^i_{jkh} + b_s\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)\right] - 2(\mathcal{B}_s\mathcal{B}_m\, \mathcal{B}_t\, y^t\, C_{irl})\left[\, a_n\, W^i_{jkh} + b_n\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)\right]$$

$- 2(\mathcal{B}_m \mathcal{B}_t y^t C_{irl})[a_{ns} W^i_{jkh} + b_{ns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p y^p(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})]$
$- 2(\mathcal{B}_s \mathcal{B}_n \mathcal{B}_t y^t C_{irl})[a_m W^i_{jkh} + b_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$- 2(\mathcal{B}_n \mathcal{B}_t y^t C_{irl})[a_{ms} W^i_{jkh} + b_{ms}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})]$
$-2(\mathcal{B}_s \mathcal{B}_t y^t C_{irl})[a_{mn} W^i_{jkh} + b_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_n\mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})]$
$- 2(\mathcal{B}_t y^t C_{irl})[a_{mns} W^i_{jkh} + b_{mns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{mn}\mathcal{B}_q y^q(\delta^i_h C_{jks} - \delta^i_k C_{jhs}) - 2\mathcal{B}_s b_m \mathcal{B}_p y^p (\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - 2b_m \mathcal{B}_p y^p \mathcal{B}_s(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})]$
$- 2(\mathcal{B}_s \mathcal{B}_n \mathcal{B}_t y^t C_{irm})[a_l W^i_{jkh} + b_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$- 2(\mathcal{B}_n \mathcal{B}_t y^t C_{irm})[a_{ls} W^i_{jkh} + b_{ls}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p y^p(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})]$
$- 2(\mathcal{B}_s \mathcal{B}_t y^t C_{irm})[a_{ln} W^i_{jkh} + b_{ln}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_n\mathcal{B}_p y^p(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})]$
$- 2(\mathcal{B}_t y^t C_{irm})[a_{lns} W^i_{jkh} + b_{lns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{ln}\mathcal{B}_q y^q(\delta^i_h C_{jks} - \delta^i_k C_{jhs}) - 2\mathcal{B}_s b_l \mathcal{B}_p y^p(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - 2b_l \mathcal{B}_p y^p \mathcal{B}_s(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})]$
$- 2(\mathcal{B}_s \mathcal{B}_t y^t C_{irn})[a_{lm} W^i_{jkh} + b_{lm}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_m\mathcal{B}_p y^p(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})]$
$- 2(\mathcal{B}_t y^t C_{irn})[a_{lms} W^i_{jkh} + b_{lms}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{ms}\mathcal{B}_q y^q - 2\mathcal{B}_l b_s \mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})$
$- 2(\mathcal{B}_t y^t C_{irs})[a_{lmn} W^i_{jkh} + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{lm}\mathcal{B}_q y^q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - 2\mathcal{B}_n b_l \mathcal{B}_p y^p (\delta^i_h C_{jkm} - \delta^i_k C_{jhm}) - 2b_l \mathcal{B}_p y^p \mathcal{B}_n(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})].$ (3.28)

Or can be written as

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C_{jrkh} = a_{lmns}C_{jrkh} + b_{lmns}(g_{rh}g_{jk} - g_{rk}g_{jh}) - 2b_{lmn}\mathcal{B}_r y^r(\delta^i_h C_{jks} - \delta^i_k C_{jhs})g_{ir}$
$- 2\mathcal{B}_s b_{lm}\mathcal{B}_q y^q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})g_{ir} - 2b_{lm}\mathcal{B}_q y^q \mathcal{B}_s(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})g_{ir}$
$- 2\mathcal{B}_s\mathcal{B}_n b_l \mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir} - 2\mathcal{B}_n b_l \mathcal{B}_p y^p \mathcal{B}_s(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir}$
$- 2\mathcal{B}_s b_l \mathcal{B}_p y^p \mathcal{B}_n(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir} - 2b_l\mathcal{B}_q y^q \mathcal{B}_s\mathcal{B}_n(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir}$
$-2(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_t y^t C_{irl})W^i_{jkh} - 2(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_t y^t C_{irl})[a_s W^i_{jkh} + b_s(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$- 2(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_t y^t C_{irl})[a_n W^i_{jkh} + b_n(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$- 2(\mathcal{B}_m\mathcal{B}_t y^t C_{irl})[a_{ns} W^i_{jkh} + b_{ns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p y^p(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})]$
$- 2(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_t y^t C_{irl})[a_m W^i_{jkh} + b_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$- 2(\mathcal{B}_n\mathcal{B}_t y^t C_{irl})[a_{ms} W^i_{jkh} + b_{ms}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})]$
$- 2(\mathcal{B}_s\mathcal{B}_t y^t C_{irl})[a_{mn} W^i_{jkh} + b_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_n\mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})]$
$- 2(\mathcal{B}_t y^t C_{irl})[a_{mns} W^i_{jkh} + b_{mns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{mn}\mathcal{B}_q y^q(\delta^i_h C_{jks} - \delta^i_k C_{jhs})$
$- 2\mathcal{B}_s b_m \mathcal{B}_p y^p(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - 2b_m\mathcal{B}_p y^p \mathcal{B}_s(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})]$
$- 2(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_t y^t C_{irm})[a_l W^i_{jkh} + b_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$- 2\mathcal{B}_n\mathcal{B}_t y^t C_{irm}[a_{ls} W^i_{jkh} + b_{ls}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p y^p(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})]$
$- 2(\mathcal{B}_s\mathcal{B}_t y^t C_{irm})[a_{ln} W^i_{jkh} + b_{ln}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_n\mathcal{B}_p y^p(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})]$
$- 2(\mathcal{B}_t y^t C_{irm})[a_{lns} W^i_{jkh} + b_{lns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{ln}\mathcal{B}_q y^q(\delta^i_h C_{jks} - \delta^i_k C_{jhs}) - 2\mathcal{B}_s b_l\mathcal{B}_p y^p (\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - 2b_l\mathcal{B}_p y^p \mathcal{B}_s(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})] - 2(\mathcal{B}_s\mathcal{B}_t y^t C_{irn})[a_{lm} W^i_{jkh} + b_{lm}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_m\mathcal{B}_p y^p(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})]$
$- 2(\mathcal{B}_t y^t C_{irn})[a_{lms} W^i_{jkh} + b_{lms}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{ms}\mathcal{B}_q y^q(\delta^i_h C_{jkl} - \delta^i_k C_{jhl}) - 2b_m\mathcal{B}_p y^p \mathcal{B}_l (\delta^i_h C_{jks} - \delta^i_k C_{jhs})] - 2(\mathcal{B}_t y^t C_{irs})[a_{lmn} W^i_{jkh} + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_{lm}\mathcal{B}_q y^q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - 2\mathcal{B}_n b_l\mathcal{B}_p y^p(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})$
$-2b_l\mathcal{B}_p y^p \mathcal{B}_n(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})] - \frac{1}{2}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l (g_{jk}R_{rh} + g_{rh}R_{jk} + g_{jh}R_{jk} - g_{jh}R_{rk} - g_{rk}R_{jh})$
$- \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l(g_{jh}g_{rk} - g_{jk}g_{rh}) - (g_{jk}R_{rh} - g_{rk}R_{jh})$
$+ \frac{1}{2}a_{lmns}(g_{jk}R_{rh} + g_{rh}R_{jk} + g_{jh}R_{jk} - g_{jh}R_{rk} - g_{rk}R_{jh}) + a_{lmns}\frac{R}{6}(g_{jh}g_{rk} - g_{jk}g_{rh}) + \frac{1}{3}a_{lmns}(g_{jk}R_{rh} - g_{rk}R_{jh})$. (3. 29)

This shows that

$\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l C_{jrkh} = a_{lmns}C_{jrkh} + b_{lmns}(g_{rh}g_{jk} - g_{rk}g_{jh}) - 2b_{lmn}\mathcal{B}_r y^r(\delta^i_h C_{jks} - \delta^i_k C_{jhs})g_{ir} - 2\mathcal{B}_s b_{lm}\mathcal{B}_q y^q(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})g_{ir} - 2b_{lm}\mathcal{B}_q y^q\mathcal{B}_s(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})g_{ir}$
$(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir} - 2\mathcal{B}_n b_l\mathcal{B}_p y^p\mathcal{B}_s(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir} - 2\mathcal{B}_s b_l\mathcal{B}_p y^p\mathcal{B}_n(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir}$
$-2b_l\mathcal{B}_q y^q \mathcal{B}_s\mathcal{B}_n(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})g_{ir}.$ (3.30)

If and only if

$2(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_t y^t C_{irl})(W^i_{jkh}) + 2(\mathcal{B}_n\mathcal{B}_m\mathcal{B}_t y^t C_{irl})[a_s W^i_{jkh} + b_s(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$+ 2(\mathcal{B}_s\mathcal{B}_m\mathcal{B}_t y^t C_{irl})[a_n W^i_{jkh} + b_n(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$
$+ 2(\mathcal{B}_m\mathcal{B}_t y^t C_{irl})[a_{ns} W^i_{jkh} + b_{ns}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2b_s\mathcal{B}_p y^p(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})]$
$+ 2(\mathcal{B}_s\mathcal{B}_n\mathcal{B}_t y^t C_{irl})[a_m W^i_{jkh} + b_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})]$

$+ 2(\mathcal{B}_n\mathcal{B}_t\, y^t\, C_{irl})\,[a_{ms}\, W^i_{jkh} + b_{ms}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2\, b_s\mathcal{B}_p\, y^p(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm})]$
$+ 2\,(\mathcal{B}_s\,\mathcal{B}_t\, y^t\, C_{irl})[\, a_{mn}\, W^i_{jkh} + b_{mn}(\,\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) - 2\, b_n\mathcal{B}_p\, y^p(\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm})]$
$+ 2\,(\mathcal{B}_t\, y^t\, C_{irl})[\, a_{mns}\, W^i_{jkh} + b_{mns}\,(\,\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2\, b_{mn}\,\mathcal{B}_q\, y^q\,(\,\delta^i_h\, C_{jks} - \delta^i_k\, C_{jhs}) - 2\,\mathcal{B}_s b_m\,\mathcal{B}_p\, y^p\,(\,\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}) - 2\, b_m\,\mathcal{B}_p\, y^p\,\mathcal{B}_s(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn})]$
$+ 2\,(\mathcal{B}_s\,\mathcal{B}_n\,\mathcal{B}_t\, y^t\, C_{irm})[\, a_l\, W^i_{jkh} + b_l(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,)]$
$+ 2\,\mathcal{B}_n\,\mathcal{B}_t\, y^t\, C_{irm}[\, a_{ls}\, W^i_{jkh} + b_{ls}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2\, b_s\mathcal{B}_p\, y^p(\,\delta^i_h\, C_{jkl} - \delta^i_k\, C_{jhl}\,)]$
$+ 2\,(\mathcal{B}_s\,\mathcal{B}_t\, y^t\, C_{irm})[\, a_{ln}\, W^i_{jkh} + b_{ln}(\,\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2\, b_n\mathcal{B}_p\, y^p(\,\delta^i_h\, C_{jkl} - \delta^i_k\, C_{jhl}\,)]$
$+ 2\,(\,\mathcal{B}_t\, y^t\, C_{irm})[\, a_{lns}\, W^i_{jkh} + b_{lns}\,(\,\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2\, b_{ln}\,\mathcal{B}_q\, y^q\,(\delta^i_h\, C_{jks} - \delta^i_k\, C_{jhs}) - 2\,\mathcal{B}_s b_l\,\mathcal{B}_p\, y^p\,(\,\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,) - 2\, b_l\,\mathcal{B}_p\, y^p\,\mathcal{B}_s(\,\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}\,)]$
$+ 2\,(\,\mathcal{B}_s\,\mathcal{B}_t\, y^t\, C_{irn})[\, a_{lm}\, W^i_{jkh} + b_{lm}\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) - 2\, b_m\mathcal{B}_p\, y^p(\delta^i_h\, C_{jkl} - \delta^i_k\, C_{jhl}\,)]$
$+ 2\,(\,\mathcal{B}_t\, y^t\, C_{irn})[\, a_{lms}\, W^i_{jkh} + b_{lms}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\,) - 2\, b_{ms}\,\mathcal{B}_q\, y^q\,(\delta^i_h\, C_{jkl} - \delta^i_k\, C_{jhl}) - 2\,\mathcal{B}_l b_s\,\mathcal{B}_p\, y^p\,(\,\delta^i_h\, C_{jkm} - \delta^i_k\, C_{jhm}\,) - 2\, b_m\,\mathcal{B}_p\, y^p\,\mathcal{B}_l(\,\delta^i_h\, C_{jks} - \delta^i_k\, C_{jhs}\,)] + 2\,(\,\mathcal{B}_t\, y^t\, C_{irs})[\, a_{lmn}\, W^i_{jkh} + b_{lmn}(\delta^i_h\, g_{jk} - \delta^i_k g_{jh}) - 2\, b_{lm}\,\mathcal{B}_q\, y^q(\delta^i_h\, C_{jkn} - \delta^i_k\, C_{jhn}) - 2\,\mathcal{B}_n b_l\,\mathcal{B}_p\, y^p\,(\,\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\,) - 2\, b_l\,\mathcal{B}_p y^p\,\mathcal{B}_n\,(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\,)] + \frac{1}{2}\,\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(\, g_{jk}R_{rh} + g_{rh}\, R_{jk} + g_{jh}R_{jk} - g_{jh}\, R_{rk} - g_{rk}\, R_{jh}) + \frac{R}{6}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(g_{jh}g_{rk} - g_{jk}g_{rh}) + \frac{1}{3}\mathcal{B}_s\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l\,(g_{jk}R_{rh} - g_{rk}R_{jh}) - \frac{1}{2}\, a_{lmns}(\, g_{jk}\, R_{rh} + g_{rh}\, R_{jk} + g_{jh}R_{jk} - g_{jh}\, R_{rk} - g_{rk}\, R_{jh}\,) - a_{lmns}\,\frac{R}{6}\,(\, g_{jh}\, g_{rk} - g_{jk}\, g_{rh}\,)$
$- \frac{1}{3} a_{lmns}\,(\, g_{jk}\, R_{rh} - g_{rk}\, R_{jh}\,) = 0. \qquad (3.31)$

Thus, the proof of theorem is completed, we get

Theorem 3.5. In GBW-FRF$_n$, associate tensor C_{jrkh} (Conformal curvature tensor C^i_{jkh}) is a generalized fourth recurrent Finsler space if the condition (3.31) holds good.

4. Conclusions and Recommendations

A generalized $\mathcal{B}$W-fourth recurrent space in Finsler space is satisfied in condition (2.1).
In G$\mathcal{B}$W-FRF$_n$, $\mathcal{B}$-covariant derivatives of the fourth orders for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by (2.2) and (2.3).
In G$\mathcal{B}$W-FRF$_n$, the condition of being necessary and sufficient for the conformal tensor C^i_{jkh} is a generalized fourth recurrent if the equation (3.8) holds. In G$\mathcal{B}$W-FRF$_n$, Ricci tensor C_{jk} is a generalized fourth recurrent if the equation (3.17) holds. In G$\mathcal{B}$W-FRF$_n$, the associate curvature tensor C_{jrkh} is a generalized 4th recurrent if the condition (3.31) holds good.
The authors argue that further study and advancement in a generalized $\mathcal{B}$W- fifth recurrent Finsler spaces is necessary and tie it in with Finsler space's distinctive space features.

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بعض المتطابقات بين مؤتر الانحناء لويلي ومؤتر الانحناء المطابق

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الملخص

في هذه الورقة البحثية، نستطلع بعض التطابقات بين مؤترانحناء ويل (ويُعرف أيضًا بمُؤتر انحناء التطابق) مؤترانحناء المطابقة. أولاً، نقدم المفاهيم الأساسية لكل من مؤترانحناء ويل ومؤتر انحناء المطابقة. ثم نستنتج بعض التطابقات بين هذين المُؤترين. وأخيرا، نطبق هذه التطابقات على بعض الأمثلة.

الكلمات المفتاحية: مشتقة بارو ارد من الرتبة الرابعة، المُؤترالأسقاطي لويلي W^i_{jkh} ، مُوتر انحناء المطابقة و المُؤترالتقوسي الثالث لكارتان R^i_{jkh}.