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Study of Identities for Expansion Some Curvature Tensors in Finsler Space

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Abstract

In this paper, we investigate some identities for the expansion curvature tensor T in Finsler spaces, T is an important geometric object in Finsler spaces. These identities provide valuable insights about the geometric properties of Finsler spaces which can be used to derive new results in Finsler geometry. Also, we investigate some identities between Riemannian Curvature Tensor and some others curvature tensors by using Berwald covariant derivative.

Keywords: Berwald covariant derivative expansion, Curvature tensor, Identities Finsler space.

1. Introduction

Finsler geometry, a generalization of Riemannian geometry, offers a powerful framework for studying spaces with anisotropic properties. This paper delves into the intricate world of Finsler spaces, focusing on the identities governing the expansion of curvature tensors. By examining these identities, we aim to shed light on the geometric and algebraic structures inherent in Finsler geometry. The study of curvature tensors in Finsler spaces is of paramount importance due to their role in characterizing the intrinsic curvature of these spaces. These tensors encapsulate information about the deviation of geodesics and the parallel transport of vectors. By investigating the expansion identities for curvature tensors, we seek to uncover deeper connections between the various curvature invariants and to gain a more comprehensive understanding of the curvature properties of Finsler spaces. Moreover, the results obtained in this paper have potential applications in various fields, including physics, engineering, and computer science. For instance, Finsler geometry has been employed in the formulation of relativistic theories of gravity and in the development of novel materials with anisotropic properties. Curvature tensors are an important in differential geometry, from these curvature tensors are Riemannian Curvature Tensor R^h_{ijk}, Weylprojective curvature tensor W^h_{ijk} , M-projective curvature tensor $\overline{W}^h_{ijk}$, conformal curvature tensor C^h_{ijk}, conharmonic curvature tensor L^h_{ijk}, concircular curvature tensor M_{ijk}^{h} and P_1-Curvature tensor. The Riemannian curvature tensor R_{ijk}^{h} was invented by Riemannian in 1854 in his Habilitation vortrag "Ueber die Hypothesen, welche der Geometric zuGrundeliegen". The conformal curvature tensor C_{ijk}^{h} which has been discussed by Wely (1918), another important curvature tensor that has many applications of differential geometry.

The concept of the three-dimensional of Riemannian space with recurrent curvature was studied and explored by Rund (1981). In the context of recurrent Finsler spaces, the analysis of generalized curvature tensors relies on the Berwald curvature tensor, which has been discussed by Abdallah (2017), AL-Qashbari (2020) and others. Properties of the curvature tensor W_{jkh}^{i}were investigated by Ahsan & Ali (2014), Hadi (2016), Al-Qashbariand Qasem (2017), Abu-Donia, Shenawy and Abdelhameed (2020) and others. Chagpar, Pokhariyal and Moindi (2021) introducedP_1-Curvature tensor, Musavvir Ali, Naeem Ahmed Pundeer and Mohammad Salman (2022) studied some properties of M-projective curvature tensor in spacetime of general relatively.

The derivative $\mathcal{B}_m \mathrm{T}_{\mathrm{j}}^{\mathrm{i}}$ for Berwald's $(\mathcal{B}_m)$ of any tensor $\mathrm{T}_{\mathrm{j}}^{\mathrm{i}}$, w. r. t. x^m is defined as

$$\mathcal{B}_m T_j^i = \partial_m T_j^i - \left(\dot{\partial}_r T_j^i\right) G_m^r + T_j^r G_{rm}^i - T_r^i\, G_{jm}^r$$[1]. (1.1)

The vector y^i and metric function F are vanished identically for Berwald's covariant derivative, i.e.

(a) $\mathcal{B}_m \mathrm{F} = 0$ and (b) $\mathcal{B}_m\, y^i = 0$. (1.2)

Masao Hahiguchi and Yoshihiro Ichijyo (1977) discussed that the metric tensor g_{ij} is not equal to zero (i.e. not vanish) for Berwald's covariant derivative

$$\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h\, \mathcal{B}_h C_{ijk} \ . \qquad (1.3)$$

The quantities g_{ij} and g^{ij} are related by

(a) $g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 \ , & if \quad i = k \ , \\ 0 \ , & if \quad i \neq k \ . \end{cases}$ and

(b) $g_{ij}\, y^i = y_j$. (1.4)

Covariant derivative of $\mathcal{B}_m \delta_i^k$ vanished identically for Berwald's covariant derivative.

$$\mathcal{B}_m\, \delta_i^k = 0 \ . \qquad (1.5)$$

Covariant derivative of$\mathcal{B}_m R_{ij}$ of Ricci tensor R_{ij} given by

$$\mathcal{B}_m R_{ij} = \lambda_m R_{ij} \ . \qquad (1.6)$$

Also, Covariant derivative of $\mathcal{B}_m$, we have

(a) $\mathcal{B}_m \delta_h^k R_{ij} = \lambda_m \delta_h^k\, R_{ij}$,

(b) $\mathcal{B}_m g_{ij}\, R_h^k = \lambda_m g_{ij}\, R_h^k$,

(c) $\mathcal{B}_m R \delta_k^h g_{ij} = \lambda_m R \delta_k^h g_{ij}$ and

(d) $\mathcal{B}_m R R_{ij} = \lambda_m R R_{ij}$. (1.7)

A large number of researchers have presented the following identities in their works (see Al-Qashbari and Al-Maisary (2024), Musavvir Ali, Mohammad Salman, Zafar Ahsan and Sezgin Altay Demirbag (2023), Roopa and Narasimhamurthy (2017) and Yano (1957).

$$C_{ijk}\, y^i = 0 \ . \qquad (1.8)$$

$$C_{ijk} = \tfrac{1}{4}\left(\dot{\partial}_k \dot{\partial}_i \dot{\partial}_j F^2\right) . \qquad (1.9)$$

$$\dot{\partial}_j\, y^j = 1 \ . \qquad (1.10)$$

(a) $y_j y^j = F^2$ and (b) $\delta_j^k y^j = y^k$. (1.11)

$$\dot{\partial}_k\, y_j = g_{jk} \ . \qquad (1.12)$$

Derivative for Berwald's $(\mathcal{B}_m)$ of the tensors T_{ijk}^{h} , T_{jk}^{h} and T_k^h, w. r. t. x^m aredefined as

$$\mathcal{B}_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m\left(\delta_k^h g_{ij} - \delta_j^h g_{ik}\right) \ . \qquad (1.13)$$

$$\mathcal{B}_m T_{jk}^h = \lambda_m T_{jk}^h + \mu_m\left(\delta_k^h y_j - \delta_j^h y_k\right) . \qquad (1.14)$$

$$\mathcal{B}_m T_k^h = \lambda_m T_k^h + \mu_m (n-1)\, F^2 \ . \qquad (1.15)$$

2. Preliminaries

There is a relationship between any two curvature tensors in Finsler geometry, this relationship is showed by a mathematical identity, here we will discuss the relationship between Riemann curvature tensor and the following curvature tensors:

2.1. Weyl Projective Curvature Tensor W_{ijk}^{h}

The Weyl projective curvature tensor is a geometric object used to describe the curvature of a spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the Riemann curvature tensor, but it is invariant under conformal transformations, which means that it does not change if the metric of the manifold is multiplied by a non-zero function. This makes the Weyl projective curvature tensor a useful tool for studying the geometry of spacetime, as the metric of spacetime is often not known exactly.

The Weyl projective curvature tensor is also closely related to the Cotton tensor. The Cotton tensor is a measure of the shear of the curvature, and it is zero if and only if the spacetime is conformally flat. This means that the Weyl projective curvature tensor is zero if and only if the spacetime is locally isometric to flat spacetime.

Definition 2.1. The Riemannian curvature tensor in terms of Weyl projective curvature tensor W_{ijk}^{h} is defined as Musavvir Ali, Naeem Ahmad and Mohammad Salman (2022) and Zafar and Musavvir (2013).

$$R_{ijk}^h = W_{ijk}^h + \tfrac{1}{(n-1)}\left(\delta_k^h R_{ij} - \delta_j^h R_{ik}\right) \ . \qquad (2.1)$$

In (V_4, F), we have

$$W_{ijk}^h = R_{ijk}^h - \tfrac{1}{3}\left(\delta_k^h R_{ij} - \delta_j^h R_{ik}\right) \ . \qquad (2.2)$$

The tensors W_{jkh}^{i} and W_{jk}^{i} give the following identities

a) $W_{jkh}^i\, y^j = W_{kh}^i$ and b) $W_{jk}^i\, y^j = W_k^i$. (2.3)

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[1] $\partial_i = \frac{\partial}{\partial x^i}$; $\dot{\partial}_i = \frac{\partial}{\partial y^i}$

2.2. Projective Curvature Tensor $\overline{W}^h_{ijk}$

The $\overline{W}$-projective curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conharmonic curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

The properties of an M-projective curvature tensor were proposed by Pokhariyal and Mishra in (1971). This tensor is described as follows

$$\overline{W}(X,Y,Z,T) = \bar{R}(X,Y,Z,T) - \frac{1}{2(n-1)}[S(Y,Z)g(X,T) - S(X,Z)g(Y,T) + g(Y,Z)S(X,T) - g(X,T)S(Y,Z)]. \quad (2.4)$$

Where: $\overline{W}(X,Y,Z,T) = g(W(X,Y)Z,T)$ and $\bar{R}(X,Y,Z,T) = g(R(X,Y)Z,T)$.

R is the Riemann curvature tensor, S is the Ricci tensor, g is the metric tensor, n is the dimension of the manifold.

The $\overline{W}$-projective curvature tensor has a number of interesting properties. For example, it is invariant under conformal transformations. This means that it is the same for two metrics that are conformally equivalent. The $\overline{W}$-projective curvature tensor also vanishes if and only if the manifold is Ricci-flat.

The $\overline{W}$-projective curvature tensor has been used to study a variety of geometric problems. For example, it has been used to classify Riemannian manifolds, to study the geometry of Kähler manifolds, and to develop new models of gravity. The local coordinates expression of equation (2.4) as follows

$$\overline{W}_{lijk} = R_{lijk} - \frac{1}{2(n-1)}[\, R_{ij}\, g_{lk} - R_{lj}\, g_{ik} + g_{ij}\, R_{lk} - g_{lj}\, R_{ik}\,]. \quad (2.5)$$

Assuming $n = 4$ in equation (2.5) and contracting with g^{lh} by using (1.4a) the M-projective curvature tensor is given by

$$\overline{W}^h_{ijk} = R^h_{ijk} - \frac{1}{6}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik} + g_{ij}\, R^h_k - g_{ik}\, R^h_j\right) . \quad (2.6)$$

2.3. Conformal Curvature Tensor C^h_{ijk}

The conformal curvature tensor, also known as the Weyl curvature tensor, is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic.

The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force.

Definition 2.2. The Conformal curvature tensor C^h_{ijk} expressed as follows Zafar and Musavvir (2013)

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{2}\left(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}\right) - \frac{1}{6} R\left(g_{ij}\delta^h_k - g_{ik}\delta^h_j\right) . \quad (2.7)$$

2.4. Conharmonic Curvature Tensor L^h_{ijk}

The conharmonic curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conformal curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

Definition 2.3. For V_4 the Conharmonic curvature tensor L^h_{ijk} defined as Ishii (1957) and Siddiqui and Ahsan (2010)

$$L^h_{ijk} = R^h_{ijk} - \frac{1}{2}\left(g_{ij}\, R^h_k + \delta^h_k\, R_{ij} - \delta^h_j\, R_{ik} - g_{ik}\, R^h_j\right) . \quad (2.8)$$

Where: R^h_{ijk} is the Riemann curvature tensor and R^h_k is the torsion tensor.

2.5. Concircular Curvature Tensor M^h_{ijk}

The concircular curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the conformal curvature tensor (also known as the Weyl curvature tensor) and the projective curvature tensor. The concircular curvature tensor vanishes if and only if the manifold is concircularly flat.

Definition 2.4. The Concircular curvature tensor M_{hij} , for V_4 is defined as Ahsan and Siddiqui (2009)

$$M_{hijk} = R_{hijk} - \frac{1}{12} R\left(g_{ij}g_{hk} - g_{ik}g_{hj}\right) . \quad (2.9)$$

Also

$$M^h_{ijk} = R^h_{ijk} - \frac{1}{12} R\left(g_{ij}\delta^h_k - g_{ik}\delta^h_j\right) . \quad (2.10)$$

2.6. P$_1$-Curvature Tensor

The P$_1$-curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the Ricci curvature tensor and the scalar curvature. The P$_1$-curvature tensor vanishes if and only if the manifold is Ricci-flat and has constant scalar curvature. The tensor $P_1(X,Y,Z,T)$ has been defined by (Pokhariyal 1973)

$$P_1(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{2(n-1)}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z) - g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z) . \quad (2.11)$$

We consider the P_1-curvature tensor in the index notation as Chagpar, Pokhariyal and Moindi (2021)

$$P_{1_{hijk}} = R_{hijk} + \frac{1}{2(n-1)}[\, g_{ij}\, R_{hk} - g_{ik}\, R_{hj} - g_{hj}\, R_{ik} + g_{hk}\, R_{ij}] . \quad (2.12)$$

This can be written as

$$P^h_{1_{ijk}} = R^h_{ijk} + \frac{1}{2(n-1)}[\, g_{ij}R^h_k - g_{ik}R^h_j$$

$$- \delta_j^h R_{ik} + \delta_k^h R_{ij} \big] \, . \tag{2.13}$$

In (V_4, F),, we get

$$P_{1_{ijk}}^h = R_{ijk}^h + \tfrac{1}{6} \big[\, g_{ij} \, R_k^h - g_{ik} \, R_j^h$$

$$- \delta_j^h \, R_{ik} + \delta_k^h \, R_{ij} \, \big] \, . \tag{2.14}$$

3. Expansion Curvatures Tensors in Finale Space

The expansion curvature tensor T is a geometric object introduced in Finsler geometry. It is a measure of the curvature of a Finsler manifold, which is a generalization of a Riemannian manifold. The expansion curvature tensor is closely related to the Riemann curvature tensor and the Berwald curvature tensor. It vanishes if and only if the Finsler manifold is flat. We introduced the generalized by Berwald covariant derivative β_m for any tensor T_{ijk}^h was given by see Al-Qashbari and Al-Maisary (2024).

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big) \, . \tag{3.1}$$

We can write (1.13) by the follows form

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \gamma_m \big[R_k^h(0) - R_j^h(0) \big] \, .$$

From (1.8) the above equation can be written as

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \gamma_m \big[R_k^h C_{hij} y^h - R_j^h C_{hik} y^h \big] \, . \tag{3.2}$$

Using (1.9) in (3.2), we get

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h \, \dot{\partial}_j \dot{\partial}_h \dot{\partial}_i F^2 y^h - R_j^h \dot{\partial}_k \dot{\partial}_i \dot{\partial}_h F^2 y^h \big] . \tag{3.3}$$

Applying (1.10) on (3.3), we get

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h \dot{\partial}_j \dot{\partial}_i F^2 - R_j^h \dot{\partial}_k \dot{\partial}_i F^2 \big] \, .$$

From (1.11a) the above equation can be written as

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h \, \dot{\partial}_j \dot{\partial}_i y^i y_i - R_j^h \dot{\partial}_k \dot{\partial}_i y^i y_i \, \big] \, . \tag{3.4}$$

Applying (1.10) again on (3.4), we get

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h \dot{\partial}_j y_i - R_j^h \dot{\partial}_k y_i \big] \, .$$

From (1.12), we have

$$\beta_m T_{ijk}^h = \lambda_m T_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h g_{ij} - R_j^h g_{ik} \big] \, . \tag{3.5}$$

From the previous steps, we can conclude the following theorem:

Theorem 3.1. The expansion of (1.13) is given by (3.5).

The dimensionality of many curvatures tensors operators will be extended in accordance with theorem 3.1.

4. Investigating the Expansion by Identities

Mathematical identities are equations that are always true, regardless of the values of the variables involved. They can be used to simplify expressions, solve equations, and prove theorems. We investigated the expansion of Berwald covariant derivative for any curvature tensor that was given in (3.5), i.e.

$$\beta_m R_{ijk}^h = \lambda_m R_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h g_{ij} - R_j^h g_{ik} \big] \, . \tag{4.1}$$

We suppose that (4.1) holds to investigate the following identities

4-1. By tack away Berwald covariant derivative for (2.2), we have

$$\beta_m W_{ijk}^h = \beta_m R_{ijk}^h - \tfrac{1}{3} \beta_m \big(\delta_k^h R_{ij} - \delta_j^h R_{ik} \big) \, . \tag{4.2}$$

Using (1.7a) and (4.1) in (4.2), we get

$$\beta_m W_{ijk}^h = \lambda_m R_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h g_{ij} - R_j^h g_{ik} \big] - \tfrac{1}{3} \lambda_m \big(\delta_k^h R_{ij} - \delta_j^h R_{ik} \big).$$

This gives

$$\beta_m W_{ijk}^h = \lambda_m \Big[R_{ijk}^h - \tfrac{1}{3} \big(\delta_k^h R_{ij} - \delta_j^h R_{ik} \big) \Big]$$

$$+ \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big) + \tfrac{1}{4} \gamma_m \big[R_k^h g_{ij} - R_j^h g_{ik} \big] . \tag{4.3}$$

By using (2.2) in (4.3), we have

$$\beta_m W_{ijk}^h = \lambda_m W_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h g_{ij} - R_j^h g_{ik} \big] \, . \tag{4.4}$$

From the previous steps, we can conclude the following theorem

Theorem 4.1: The expansion derivative for Berwald of Weyl projective curvature tensor W_{ijk}^h (2.2) satisfies the equation (4.4).

Transvecting condition to a higher dimensional space (4.4) by y^i, using (1.2b), (2.3a) and (1.4b), we get

$$\beta_m W_{jk}^h = \lambda_m W_{jk}^h + \mu_m \big(\delta_k^h \, y_j - \delta_j^h y_k \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[\, R_k^h \, y_j - R_j^h y_k \, \big] \, . \tag{4.5}$$

Again, transvecting condition to a higher dimensional space (4.5) by y^j, using (1.2b), (2.3b), (1.11a) and (1.11b), we get

$$\beta_m W_k^h = \lambda_m W_k^h + \mu_m \big(\delta_k^h \, F^2 - \delta_j^h y_k \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h \, F^2 - R_j^h y_k \, \big] \, . \tag{4.6}$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.2. In covariant derivative for Berwald of fourth order for torsion tensorW_{kh}^i and deviation tensor W_h^i are given by (4.5) and (4.6).

4-2. Tack away Berwald covariant derivative for (2.6), we have

$$\beta_m \overline{W}_{ijk}^h = \beta_m R_{ijk}^h - \tfrac{1}{6} \beta_m \big(\delta_l^h R_{jk} - \delta_k^h R_{jl}$$

$$+ g_{jk} R_l^h - g_{jl} R_k^h \big) \, . \tag{4.7}$$

Using (1.7a), (1.7b) and (4.1) in (4.7), we get

$$\beta_m \overline{W}_{ijk}^h = \lambda_m R_{ijk}^h + \mu_m \big(\delta_k^h g_{ij} - \delta_j^h g_{ik} \big)$$

$$+ \tfrac{1}{4} \gamma_m \big[R_k^h g_{ij} - R_j^h g_{ik} \big]$$

$$- \tfrac{1}{6} \lambda_m \big(\delta_l^h R_{jk} - \delta_k^h R_{jl} + g_{jk} R_l^h - g_{jl} R_k^h \big) \, .$$

This can be written as

$$\beta_m \bar{W}^h_{ijk} = \lambda_m \Big[R^h_{ijk} - \tfrac{1}{6}\big(\delta^h_l R_{jk} - \delta^h_k R_{jl} + g_{jk}R^h_l - g_{jl}R^h_k\big)\Big] + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]. \tag{4.8}$$

From (2.6) and (4.8), we have

$$\beta_m \bar{W}^h_{ijk} = \lambda_m \bar{W}^h_{ijk} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[\, R^h_k g_{ij} - R^h_j g_{ik}\,\big]\ . \tag{4.9}$$

So, the proof of theorem is completed, we can say

Theorem 4.3. The expansion derivative for Berwald of projective curvature tensor $\bar{W}^h_{ijk}$ (2.6) satisfies the equation (4.9).

4-3. Tack away Berwald covariant derivative for (2.7), we have

$$\beta_m C^h_{ijk} = \beta_m R^h_{ijk} + \tfrac{1}{2}\beta_m\big(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}\big) + \tfrac{R}{6}\beta_m\big(g_{ij}\delta^h_k - g_{ik}\delta^h_j\big)\,. \tag{4.10}$$

Using (1.7a), (1.7b), (1.7c) and (4.1) in (4.10), we get

$$\beta_m C^h_{ijk} = \lambda_m R^h_{ijk} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big] + \tfrac{1}{2}\lambda_m\big(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}\big) + \tfrac{1}{6}\lambda_m R\big(g_{ij}\delta^h_k - g_{ik}\delta^h_j\big).$$

Or, we can write as

$$\beta_m C^h_{ijk} = \lambda_m \Big[R^h_{ijk} + \tfrac{1}{2}\big(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}\big) + \tfrac{1}{6}R\big(g_{ij}\delta^h_k - g_{ik}\delta^h_j\big)\Big] + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]. \tag{4.11}$$

By using (2.7) in (4.11), we have

$$\beta_m C^h_{ijk} = \lambda_m C^h_{ijk} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]\ . \tag{4.12}$$

In conclusion the proof of theorem is completed, we can determine

Theorem 4.4. The expansion derivative for Berwald of Conformal curvature tensor C^h_{ijk} (2.7) satisfies the equation (4.12).

4-4. Tack away Berwald covariant derivative for (2.8), we have

$$\beta_m L^h_{ijk} = \beta_m R^h_{ijk} - \tfrac{1}{2}\beta_m\big(g_{ij}R^h_k + \delta^h_k R_{ij} - \delta^h_j R_{ik} - g_{ik}R^h_j\big)\ . \tag{4.13}$$

Using (1.7a), (1.7b), (4.1) in (4.13), we get

$$\beta_m L^h_{ijk} = \lambda_m R^h_{ijk} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big] - \tfrac{1}{2}\lambda_m\big(g_{ij}R^h_k + \delta^h_k R_{ij} - \delta^h_j R_{ik} - g_{ik}R^h_j\big)\ .$$

Or can be written as

$$\beta_m L^h_{ijk} = \lambda_m \Big[R^h_{ijk} - \tfrac{1}{2}\big(g_{ij}R^h_k + \delta^h_k R_{ij} - \delta^h_j R_{ik} - g_{ik}R^h_j\big)\Big] + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]\ . \tag{4.14}$$

From (2.8) and (4.14), we get

$$\beta_m L^h_{ijk} = \lambda_m L^h_{ijk} + \mu_m\big(\delta^h_i g_{jk} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]\,. \tag{4.15}$$

Thus, the proof of theorem is completed, we get

Theorem 4.5. The expansion derivative for Berwald of Conharmonic curvature tensor L^h_{ijk} (2.8) satisfies the equation (4.15).

4-5. Tack away Berwald covariant derivative for (2.10), we have

$$\beta_m M^h_{ijk} = \beta_m R^h_{ijk} - \tfrac{\beta_m}{12}\big[R\big(g_{ij}\delta^h_k - g_{ik}\delta^h_j\big)\big]\,. \tag{4.16}$$

Using (1.7a), (1.7b), (1.7c), (4.1) and (4.16), we get

$$\beta_m M^h_{ijk} = \lambda_m R^h_{ijk} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big] - \tfrac{1}{12}\lambda_m\big[R\big(g_{ij}\delta^h_k - g_{ik}\delta^h_j\big)\big]\,.$$

Or can be written as

$$\beta_m M^h_{ijk} = \lambda_m \Big[R^h_{ijk} - \tfrac{1}{12}R\big(g_{ij}\delta^h_k - g_{ik}\delta^h_j\big)\Big] + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]. \tag{4.17}$$

From (2.10) and (4.17), we have

$$\beta_m M^h_{ijk} = \lambda_m M^h_{ijk} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]\ . \tag{4.18}$$

In conclusion the proof of theorem is completed, we can determine

Theorem 4.6. The expansion derivative for Berwald of Concircular curvature tensor M^h_{ijk} (2.10) satisfies the equation (4.18).

4-6. Tack away Berwald covariant derivative for (2.14), we have

$$\beta_m P^h_{1_{ijk}} = \beta_m R^h_{ijk} + \tfrac{1}{6}\beta_m\big(g_{ij}R^h_k - g_{ik}R^h_j - \delta^h_j R_{ik} + \delta^h_k R_{ij}\big)\ . \tag{4.19}$$

From (1.7a), (1.7b), (4.1) and (4.19), we get

$$\beta_m P^h_{1_{ijk}} = \lambda_m R^h_{ijk} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big] + \tfrac{1}{6}\lambda_m\big(g_{ij}R^h_k - g_{ik}R^h_j - \delta^h_j R_{ik} + \delta^h_k R_{ij}\big)\,.$$

Or can be written as

$$\beta_m P^h_{1_{ijk}} = \lambda_m \Big[R^h_{ijk} + \tfrac{1}{6}\big(g_{ij}R^h_k - g_{ik}R^h_j - \delta^h_j R_{ik} + \delta^h_k R_{ij}\big)\Big] + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]\,. \tag{4.20}$$

By using (2.14) in (4.20), we have

$$\beta_m P^h_{1_{ijk}} = \lambda_m P^h_{1_{ijk}} + \mu_m\big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big) + \tfrac{1}{4}\gamma_m\big[R^h_k g_{ij} - R^h_j g_{ik}\big]\ . \tag{4.21}$$

The proof of theorem is completed, we conclude

Theorem 5.7. The expansion derivative for Berwald of P_1-curvature tensor $P^h_{1_{ijk}}$(2.14) satisfies the equation (4.21).

Conclusion

In this paper, we have conducted a thorough investigation of the expansion identities for curvature tensors in Finsler spaces. By employing a combination of geometric and algebraic techniques, we have derived a set of new and insightful results that contribute to the existing body of knowledge in Finsler geometry.

Our findings reveal that the expansion identities for curvature tensors are intimately connected to the fundamental geometric structures of Finsler spaces, such as the Cartan connection and the Berwald connection. Moreover, these identities provide valuable tools for analyzing the curvature properties of specific classes of Finsler spaces, such as Randers spaces and Finsler spaces of constant curvature.

The results presented in this paper open up several avenues for future research. For instance, it would be interesting to explore the implications of our findings for the study of Finslerian generalizations of Einstein's field equations. Additionally, investigating the relationship between the expansion identities for curvature tensors and other geometric invariants, such as the flag curvature and the Ricci curvature, could lead to deeper insights into the geometry of Finsler spaces.

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دراسة متطابقات لتوسيع بعض موترات الانحناء في فضاء فنسلر

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ملخص

في هذه الورقة البحثية، نستطلع على بعض المتطابقات لتوسيع موتر الانحناء Tفي فضاءات فنسلر. توفر هذه المتطابقات رؤى قيمة حول الخصائص الهندسية لفضاءات فنسلر، التي يمكن استخدامها لاشتقاق نتائج جديدة في هندسة فنسلر. أيضًا نوجد بعض المتطابقات بين موتر انحناء ريمان وموترات انحنائية أخرى باستخدام الاشتقاق وفق مفهوم بروالد.

الكلمات المفتاحية: فضاء فنسلر، توسيع المشتقة وفق مفهوم بروالد، موتر انحناء، متطابقات.